

On the accuracy of the Chakrabarti-Hudson approximation to π

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Abstract

We obtain rigorous upper and lower bounds for the error in the recent approximation for π proposed by CHAKRABARTI & HUDSON.

1 Introduction

In 2003, CHAKRABARTI and HUDSON [1] proposed a new formula in the spirit of ARCHIMEDES for the computation of π , namely

$$\pi \approx \frac{1}{30} \{32\pi_n + 4\Pi_{2n} - 6a_n\}, \quad (1.1)$$

where π_n is the perimeter of an inscribed regular polygon of n sides in a circle of perimeter π , Π_n is the perimeter of a circumscribed regular polygon of n sides in a circle of perimeter π , and a_n is the area of an inscribed regular polygon of n sides in a circle of *area* π .

Archimedes, in *The Measurement of the Circle* (MC), see [2], obtains his own approximation by starting with the (evident) inequality

$$\pi_n < \pi < \Pi_n, \quad (1.2)$$

and takes $n = 6, 12, 24, 48, 96$, respectively. He performs a brilliant *tour de force* of manipulating and rounding the inequalities resulting from continued fraction expansions of the square roots which arise in the computations of π_n and Π_n , whence he obtains his justly famous bounds

$$3\frac{10}{71} < \pi < 3\frac{1}{7}. \quad (1.3)$$

It is generally recognized, see [6], that the extant MC is a post-Archimedean revision of Archimedes' original and far more comprehensive treatment of the circle. For example, HERON, *Metrica* I, 32 [3], quotes a theorem of Archimedes from (MC), which is *not present*

in the extant version. The theorem states that *the area of a circular sector exceeds four-thirds the area of the greatest inscribable triangle*. In terms of the notation above, this inequality affirms that:

$$\pi > \frac{1}{3}(4\pi_{2n} - \pi_n). \quad (1.4)$$

The right hand side of (1.4) is a *convex combination* of π_{2n} and π_n since $\frac{4}{3} - \frac{1}{3} = 1$. Moreover, its form argues for the fact that the original treatise of Archimedes studied convergence-improvement inequalities derived by means of geometric theory. Unfortunately, such theoretical aspects apparently proved too subtle for the uses of later commentators such as Heron.

Observe that the formula of CHAKRABARTI & HUDSON (1.1), too, is a convex combination of its terms.

Such convex combinations did not reappear until the sixteenth and seventeenth centuries in the work of SNELL [7] and HUYGENS [5]. The former stated and the latter proved the following upper bound inequality:

$$\pi < \frac{2}{3}\pi_n + \frac{1}{3}\Pi_n, \quad (1.5)$$

which is a beautiful generalization of Archimedes' original inequality. Both Snell and Huygens use it in the form

$$\pi \approx \frac{2}{3}\pi_n + \frac{1}{3}\Pi_n \quad (1.6)$$

and they justifiably extol the convergence-rate improvement produced by this convex combinations of quantities already computed.

Now, Chakrabarti and Hudson ask what happens if we use the convex combination

$$\pi \approx \frac{2}{3}a_n + \frac{1}{3}A_n \quad (1.7)$$

to approximate π , where A_n is the area of a circumscribed regular polygon of n sides in a circle of area π . They prove the very interesting result that

$$\lim_{n \rightarrow \infty} \frac{\frac{2}{3}\pi_n + \frac{1}{3}\Pi_n - \pi}{\frac{2}{3}a_n + \frac{1}{3}A_n - \pi} = \frac{3}{8}, \quad (1.8)$$

which shows that *the areas converge much more slowly than the perimeters to π* .

Then they suppress the limit notation, treat (1.8) *as an equation for π* and when they reduce it algebraically they obtain the approximative formula (1.1). Passing to trigonometric functions, (1.1) becomes:

$$\pi \approx \frac{n}{30} \left\{ 32 \sin\left(\frac{\pi}{n}\right) + 4 \tan\left(\frac{\pi}{n}\right) - 3 \sin\left(\frac{2\pi}{n}\right) \right\} \equiv \Pi(n). \quad (1.9)$$

Finally, the authors offer some numerical studies of the accuracy of (1.9).

The investigation of the authors needs to be completed in several areas.

- They do not state whether the approximation (1.9) is in excess or in defect... in fact, we will prove that it is in *excess*.

- They do not develop any rigorous error analysis... no upper bounds for the error nor lower bounds; we will present such bounds.
- They offer a non-standard definition of accuracy, which they call “precision”. This makes it difficult to compare their numerical results with standard error studies. We will present the standard definition of “correct significant digits” and reformulate the discussion of the accuracy of (1.9) in light of our error bounds.

2 Error Analysis

We will prove the following theorem:

Theorem 1. *If $n \geq 32$ then the following inequality is valid:*

$$\boxed{\frac{1}{105} \left(\frac{\pi}{n}\right)^6 < \frac{\Pi(n) - \pi}{\pi} < \frac{1}{104\frac{7}{10}} \left(\frac{\pi}{n}\right)^6} \quad (2.1)$$

where the lower-bound constant $\frac{1}{105}$ is the best possible.

An immediate consequence is:

Corollary 1. *The approximation $\pi \approx \Pi(n)$ is in **excess**.* □

The inequality (2.1) bounds the *relative error* in the approximation $\Pi(n) \approx \pi$. It is well known that an approximation has n *correct significant digits* iff the relative error does not exceed $\frac{(\frac{1}{2})}{10^n}$, see [4]. Therefore, we can say:

Corollary 2. *The approximation $\pi \approx \Pi(n)$ has about $(6 \log_{10} n - 1.27)$ correct significant digits.* □

Now we turn to the proof of (2.1). We will use the MACLAURIN expansions of the functions involved. Define:

$$f(x) := \frac{\pi}{30x} (32 \sin x + 4 \tan x - 3 \sin 2x). \quad (2.2)$$

Then we see that

$$\Pi(n) = f\left(\frac{\pi}{n}\right). \quad (2.3)$$

Since the sum of a convergent alternating series is bracketed by two consecutive partial sums, we obtain:

Lemma 1. *The following inequality is valid for all $x \geq 0$:*

$$32 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \right) < 32 \sin x < 32 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \right). \quad (2.4)$$

□

The MACLAURIN expansion of the tangent function is not an alternating series. But we can still use the LAGRANGE form of the remainder to obtain:

Lemma 2. *The following inequality is valid for $0 \leq x \leq \frac{\pi}{32}$:*

$$4 \left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 \right) < 4 \tan x \\ < 4 \left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \frac{1}{85}x^{11} \right). \quad (2.5)$$

Proof. The MACLAURIN expansion of order 11 of the tangent function is

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \frac{1}{11!}R(\theta_{11})x^{11}, \quad (2.6)$$

where

$$R(x) := \frac{d^{11}}{dx^{11}}(\tan x) = 256(\tan^2 x + 1)(155925 \tan^{10} x + 467775 \tan^8 x + 509355 \tan^6 x \\ + 238425 \tan^4 x + 42306 \tan^2 x + 1382)$$

and $0 \leq \theta_{11} \leq \frac{\pi}{32}$. The function $R(x)$ monotonically increases in the interval, whence

$$353792 = R(0) < R(\theta_{11}) \leq R\left(\frac{\pi}{32}\right) = 469223.9941 \dots \quad (2.7)$$

and we conclude (since $x \geq 0$) that

$$0 \leq \frac{1}{11!}R(\theta_{11})x^{11} \leq \frac{1}{11!} \cdot 469223.9941 \dots x^{11} = \frac{1}{85.06 \dots} x^{11} < \frac{1}{85} x^{11}. \quad (2.8)$$

This completes the proof. □

Finally, as in the inequality (2.4) we obtain:

Lemma 3. *The following inequality is valid for all $x \geq 0$:*

$$-3 \left\{ 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \frac{(2x)^9}{9!} \right\} < -3 \sin 2x \\ < -3 \left\{ 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \frac{(2x)^9}{9!} - \frac{(2x)^{11}}{11!} \right\}. \quad (2.9)$$

□

Now that we have set up the technical inequalities necessary in our main proof, we enter into its details.

Proof of the main theorem. If we substitute the inequalities (2.4), (2.5) and (2.9) into the formula for $f(x)$, (2.2), we obtain

$$\frac{1}{105}x^6 + \frac{1}{360}x^8 + \frac{2776}{2338875}x^{10} < \frac{f(x)}{\pi} - 1 < \frac{1}{105}x^6 + \frac{1}{360}x^8 + \frac{10531}{26507250}x^{10}. \quad (2.10)$$

Now, the left-hand side

$$\frac{1}{105}x^6 + \frac{1}{360}x^8 + \frac{2776}{2338875}x^{10} = \frac{1}{105}x^6 \left\{ 1 + \frac{7}{24}x^2 + \frac{2776}{22275}x^4 \right\} > \frac{1}{105}x^6 \quad (2.11)$$

for all positive x . So, we have proven the lower bound (2.1) and that the constant $\frac{1}{105}$ cannot be replaced by a bigger one, i.e., *it is the best possible*.

For the upper bound, we observe that for $0 \leq x \leq \frac{\pi}{32}$,

$$\begin{aligned} \frac{1}{105}x^6 + \frac{1}{360}x^8 + \frac{10531}{26507250}x^{10} &= \frac{1}{105}x^6 \left\{ 1 + \frac{7}{24}x^2 + \frac{10531}{252450}x^4 \right\} \\ &< \frac{1}{105}x^6 \left\{ 1 + \frac{7}{24} \left(\frac{\pi}{32} \right)^2 + \frac{10531}{252450} \left(\frac{\pi}{32} \right)^4 \right\} \\ &= \frac{1}{104.705\dots}x^6 < \frac{1}{104\frac{7}{10}}x^6, \end{aligned}$$

which is the upper bound presented in (2.1).

This, together with (2.3) complete the proof of the theorem. \square

3 Numerical Error Studies

Chakrabarti and Hudson present a table of numerical studies of the error in their approximative formula (1.8). They define: a number α has *precision* n if

$$|\alpha - \pi| < \frac{1}{10^n}. \quad (3.1)$$

We already pointed out the standard definition (see [4]): an approximation \overline{N} approximates the true value N with n *correct significant digits* if the positive relative error

$$\frac{|N - \overline{N}|}{N} < \frac{(\frac{1}{2})}{10^n}. \quad (3.2)$$

Of course the two definitions will coincide sometimes, and sometimes not. However, we note that

$$|\alpha - \pi| < \frac{1}{10^n} \iff \frac{|\alpha - \pi|}{\pi} < \frac{1}{\pi 10^n} < \frac{(\frac{1}{2})}{10^n}$$

so that *an approximation with precision n is always correct to n significant digits*. It is sufficient, but not necessary, since if

$$\frac{1}{\pi 10^n} < \frac{|\alpha - \pi|}{\pi} < \frac{(\frac{1}{2})}{10^n},$$

then α will have precision $n - 1$ but still have n correct significant digits.

4 Earlier approximations

We can rewrite the approximation $1 \approx \frac{f(x)}{\pi}$ as follows:

$$\frac{\sin x}{x} \approx \frac{15 \cos x}{2 + 16 \cos x - 3 \cos^2 x}, \quad (4.1)$$

where we know that the approximate value is *smaller* than the true value by about $\frac{x^6}{105}$.

Now, $\frac{\sin x}{x}$ has the following *continued fraction expansion*:

$$\frac{\sin x}{x} = 1 - \frac{\frac{1 \cdot 2}{1 \cdot 3} \sin^2 \frac{x}{2}}{1 - \frac{\frac{1 \cdot 2}{3 \cdot 5} \sin^2 \frac{x}{2}}{1 - \frac{\frac{3 \cdot 4}{5 \cdot 7} \sin^2 \frac{x}{2}}{1 - \frac{\frac{3 \cdot 4}{7 \cdot 9} \sin^2 \frac{x}{2}}{1 - \dots}}}} \quad (4.2)$$

The first few convergents are:

$$\begin{array}{ll} \frac{p_1}{q_1} = \frac{2 + \cos x}{3}; & \text{error} = -\frac{1}{180}x^4 + \dots \quad (\text{SNELL}) \\ \frac{p_2}{q_2} = \frac{9 + 6 \cos x}{14 + \cos x}; & \text{error} = -\frac{1}{2100}x^6 + \dots \quad (\text{NEWTON}) \\ \frac{p_3}{q_3} = \frac{51 + 48 \cos x + 6 \cos^2 x}{80 + 25 \cos x}; & \text{error} = -\frac{1}{44100}x^8 + \dots \end{array} \quad (4.3)$$

where we have used $\sin^2 \frac{x}{2} = \frac{1}{2}(1 - \cos x)$. (For all these results, see [8].) Every one of these approximations is *larger* than the true value and is the *best possible*.

The Chakrabarti–Hudson approximation (4.1) has an error about twenty times greater than that of Newton’s formula, although both are of order $o(x^6)$, nor is not as simple. Indeed, it has the formal complexity of the *third* convergent without the latter’s extraordinary accuracy. Nevertheless, it is the only approximation in *defect*, its accuracy is still quite good, and it is interesting that such an *ad hoc* derivation produced such an intriguing approximation.

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